Robust LQG Controller Design for Lightly Damped Uncertain Linear Systems

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(*Received December 21, 1996*)

In this note, we consider lightly damped uncertain linear systems with natural frequency variations. This type of uncertainty has multiple uncertain parameters with multiple rank structure. It is well known that the conventional LQG or LQG/LTR methods can not be applied to such systems because of the instability and the design conservatism. To overcome such shortcomings, we propose a systematic method to design robust LQG controllers. The proposed method requires only the LQG tuning parameters and the structure information of uncertainty. It will be shown that our approach can be effectively applied to flexible structure control design problems.

Key Words: Lightly Damped Uncertain Systems, Robust LQG Controller, Real Parameter Uncertainty, Quadratic Stability, General I/O Decomposition

1. Introduction

In the active vibration control, one of the most important issues has been the robustness problem to the natural frequency variation, specially in the lightly damped systems. For example, the conventional LQG control method cannot be applied to the flexible structures such as flexible beam or plate systems because of the poor robustness to the system parameter variations. LQG/LTR method can be an alternative for the robust design method. However, it is well known that LQG/LTR generates conservative controllers to obtain the desired robustness property. Recently, H_∞ control theory is frequently applied to the robust compensator design (Doyle *et al.*, 1989; Zhou *et al.*, 1996). However, the robust performance in the time domain can not be considered by the theory (Zhou *et al.*, 1994). A notable method to deal with the robust performance in the time domain is the quadratic stabilization technique in view of the robust H_2 control theory (e. g. see Bernstein and Haddad, 1989; Petersen, 1995). In the references, the authors have defined an auxiliary quadratic cost index, which is the upper bound of the quadratic cost index of interest, and have minimized the auxiliary performance. Such frameworks are known to be systematic in formulating the robust performance in the time domain. However, previous results are limited to the structured one-block uncertainty. Therefore, the previous works can not be employed for treating the real parameter uncertainty which has multiple uncertain parameters with multiple rank structure. In this note, we investigate a systematic design method to deal with the real parameter uncertainty for the control of lightly damped flexible structures. By considering the general input-output (I/O) decomposition of real parameter uncertainty (Kim, 1995; Kim and Park, 1995), we extend the Bernstein and Haddads Riccati approach. A design example will be given to show the effectiveness of the proposed approach.

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2. Preliminaries

We use the following definition of the quadratic stability for considering the robust stability.

Definition 2.1 The system, $\dot{x}=f(t,x)$, is *quadratically stable* if there exists a quadratic Lyapunov function such that $V=x^TPx$ for a positive definite matrix P. ■

It is noted that the quadratic stability, which implies the asymptotic stability of the system, plays an important role to describe the robust stability of uncertain systems. Recently, the relation between the small gain theorem and the quadratic stability is known as given in the following lemma.

Lemma 2.1 (Khargonekar et al., 1990) Consider the uncertain system given by

$$\dot{x}=[A_0+\Delta A(t)]x,\quad \Delta A(t)=DF(t)E \quad (1)$$

where D and F are the known constant matrices, and $F(\bullet)$ is a time-varying uncertain parameter matrix satisfying $F(t)^TF(t)\leq I$ for all $t\in\Re^+$. Then, The system (1) is quadratically stable for all $F(t): F(t)^TF(t)\leq I\,\forall t$ if and only if the following conditions hold:

(i) A_o is asymptotically stable.

(ii) $\|E(j\omega I-A_o)^{-1}D\|_\infty<1$. ■

3. Robust LQG Controller

Consider the uncertain system described by

$$\dot{x}=(A+\Delta A(t))x+Bu+Fw \quad (2a)$$

$$y=Cx+v \quad (2b)$$

where $x\in\Re^n$, $u\in\Re^m$, $y\in\Re^l$, $E[w(t)w(\tau)^T]=W\delta(t-\tau)$, $E[v(t)v(\tau)^T]=V\delta(t-\tau)$, $W\geq 0$, $V>0$ for uncorrelated noises. The uncertainty is the real parameter uncertainty which may be time-varying as follows:

$$\Delta A(t)=\sum_{i=1}^{r}\delta_i(t)E_i,\quad |\delta_i(t)|\leq 1 \quad (3)$$

where E_i is the known constant matrix with *rank* $(E_i)=q_i$ and $\delta_i(\cdot)$ is a Lebesgue-measurable scalar function. Note that the uncertainty in the natural frequencies of the lightly damped flexible structure can be effectively described by the real parameter uncertainty defined in (3). An illustrative example is given in the following.

Example Consider the stiffness matrix such as $K=\begin{bmatrix} k_1 & k_1+k_2 \\ k_1+k_2 & k_2\end{bmatrix}$. In this case, the uncertainty can be expressed as $\Delta K=\delta k_1\cdot\begin{bmatrix}1 & 1\\ 1 & 0\end{bmatrix}+\delta k_2\cdot\begin{bmatrix}0 & 1\\ 1 & 1\end{bmatrix}$ in the form of (3).

It is known that the uncertainty in (3) has the infinite number of I/O decomposition form as follows:

Definition 3.1 (Kim and Park, 1995) Let us define a set

$$S_\Delta^\Gamma\equiv\{\Gamma\mid\Gamma=blockdiag[\Gamma_1,\cdots,\Gamma_r],\ det(\Gamma_i)\neq 0,\ \Gamma_i\in\Re^{q_i\times q_i}\} \quad (4a)$$

Then, given the uncertainty, it is always possible to construct the I/O decomposition form as follows:

$$\Delta A(t)=(M\Gamma)\Delta(t)(\Gamma^{-1}N)\quad \text{for any } \Gamma\in S_\Delta^\Gamma \quad (4b)$$

where

$$M\equiv[M_1,\cdots,M_r],$$
$$N^T\equiv[N_1^T,\cdots,N_r^T],$$
$$\Delta(t)\equiv\begin{bmatrix}\delta_1(t)I_{q_1\times q_1} & & \\ & \vdots & \\ & & \delta_r(t)I_{q_r\times q_r}\end{bmatrix}\in\Re^{h\times h}$$

for the minimal rank decomposition of E_i such that $E_i=M_iN_i$, $rank(M_i)=rank(N_i)=q_i$. In this case, (4) is said to be the general I/O decomposition of (3).

The purpose of Definition 3.1 is to express all the possible I/O decomposition of the uncertainty in (3) by a design variable Γ. Because of introducing a scaling matrix variable, nonuniqueness of M_i and N_i in the minimal rank decomposition is not a concern, any more.

Without loss of generality, we follow all the basic assumptions on the nominal system as in the standard LQG theory. By using the observer

-based controller

$$\dot{\hat{x}} = A\hat{x} + Bu + K_f(y - C\hat{x}), \quad u = -K_c\hat{x} \quad (5)$$

where $K_c \in \Re^{m\times n}$, $K_f \in \Re^{n\times l}$, the closed loop can be written as follows:

$$\dot{x}_e = (A_e + \Delta A_e)x_e + F_e w_e \quad (6)$$

where

$$x_e = \begin{bmatrix} x \\ x-\hat{x} \end{bmatrix}, \; w_e = \begin{bmatrix} w \\ v \end{bmatrix},$$

$$A_e = \begin{bmatrix} A - BK_c & BK_c \\ 0 & A - K_f C \end{bmatrix},$$

$$\Delta A_e = \begin{bmatrix} \Delta A & 0 \\ \Delta A & 0 \end{bmatrix}, \; F_e = \begin{bmatrix} F & 0 \\ F & -K_f \end{bmatrix},$$

$E[w_e(t)w_e^T(\tau)] = W_e\delta(t-\tau)$ for $W_e = \begin{bmatrix} W & 0 \\ 0 & V \end{bmatrix}$. It is noted that the augmented uncertainty has the general I/O decomposition as follows:

$$\Delta A_e = (M_e\Gamma)\Delta(t)(\Gamma^{-1}N_e) \text{ for any } \Gamma \in S_\Delta^\Gamma \quad (7)$$

where $M_e^T = [M^T, M^T]$, $N_e = [N, 0_{h\times 2n}]$. From (6) and (7), we can derive the following lemma.

Lemma 3.1 The system in (6) is quadratically stable if the following conditions hold:

(i) A_e is asymptotically stable.

(ii) $\| \Gamma^{-1}N_e(j\omega I - A_e)^{-1}M_e\Gamma \|_\infty < 1$ for some $\Gamma \in S_\Delta^\Gamma$.

(Proof) It is evident from Lemma 2.1. The necessity cannot be proven because of the diagonal structure of uncertainty $\Delta(\cdot)$. (Q. E. D)

Note that Lemma 3.1 is a block-diagonally scaled small gain condition. If we do not adopt the general I/O decomposition, the quadratic stability becomes very conservative for the multiple uncertain parameters with multiple rank structure. The importance of the general I/O decomposition in the robust full state feedback problem has been pointed out in some materials (Kim, 1995; Savkin and Petersen, 1995).

Now, we consider the stationary LQG performance index as follows:

$$J_{\Delta A} \equiv \min_{K_c, K_f} \lim_{t_f\to\infty} E\left[\frac{1}{t_f}\int_0^{t_f}(x^TQx + u^TRu)\,dt\right] \text{ for a given } \Delta A \quad (8)$$

where $Q \geq 0$, $R > 0$. For the structured one-block uncertainty, Bernstein and Haddad (1988) has shown that $J_{\Delta A}$ is bounded as follows:

$$J_{\Delta A} \leq \min_{K_c, K_f} tr[F_e W_e F_e^T P] \text{ for any admissible } \Delta A \quad (9)$$

where $0 \leq P \in \Re^{2n\times 2n}$ and P satisfies

$$A_e^T P + PA_e + Q_e + \Psi(P) = 0 \quad (10)$$

for $Q_e = \begin{bmatrix} Q + K_c^T R K_c & -K_c^T R K_c \\ -K_c^T R K_c & K_c^T R K_c \end{bmatrix}$ and any constant function $\nu(\cdot)$ such that $\Delta A_e^T P + P\Delta A_e \leq \Psi(P)$. We propose the following theorem to extend the Bernstein and Haddads Riccati approach (1988, 1989) for treating the real parameter uncertainty.

Theorem 3.1 The given system (6) is quadratically stable if there exist a symmetric matrix $P \in \Re^{2n\times 2n}$ and an $X \in \{\Gamma\Gamma^T \mid \Gamma \in S_\Delta^\Gamma\}$, for a $K_c \in \Re^{m\times n}$ and a $K_f \in \Re^{n\times l}$, satisfying

$$A_e^T P + PA_e + Q_e + N_e^T X^{-1} N_e + PM_e X M_e^T P = 0 \quad (11)$$

such that $A_e + M_e X M_e^T P$ is asymptotically stable. Further more, for all the admissible uncertainty,

$$J_{\Delta A} \leq \bar{J} \equiv \min_{X, K_c, K_f} tr[F_e W_e F_e^T P] \quad (12)$$

(Proof) Since $A_e + M_e X M_e^T P$ is asymptotically stable, so is A_e. By the bounded real lemma (Green and Limebeer, 1995), (11) can be equivalently written as $\left\| \begin{bmatrix} X^{-1/2}N_e \\ Q_e^{-1/2} \end{bmatrix} (j\omega I - A_e)^{-1} M_e X^{1/2} \right\|_\infty < 1$. It implies that $\| X^{-1/2}N_e(j\omega I - A_e)^{-1}M_e X^{1/2} \|_\infty < 1$. Since $X^{1/2}$ is a real positive definite matrix, there always exists a $\Gamma \in S_\Delta^\Gamma$ and a real orthogonal matrix U such that $X^{1/2} = \Gamma U$, $UU^T = I$. Therefore, by the invariance of H_∞-norm for the unitary transformation, the inequality is identical to the condition (ii) in Lemma 3.1. Also, the upper bound of cost index can be obtained by applying Bernstein and Haddads results described in (9) and (10) after showing

$$\begin{aligned} \Delta A_e^T P + P\Delta A_e &= (N_e^T\Gamma^{-T})\Delta(\Gamma^T M_e^T P) \\ &\quad + (PM_e^T\Gamma)\Delta(\Gamma^{-1}N_e) \\ &\leq N_e^T X^{-1} N_e + PM_e X M_e^T P \\ &= \Psi(P) \quad (13) \end{aligned}$$

(Q. E. D)

Based on Theorem 3.1, we define an optimization problem for the robust LQG controller as follows:

Definition 3.2 Consider the pair of gains, $(K_c^{RLQG},\ K_f^{RLQG}) = arg \min_{X, K_c, K_f} tr[F_e W_e F_e^T P]$, where $arg(\bullet)$ denotes the optimal values of variables. Then, the observer based-controller in (5) with the gains (K_c^{RLQG}, K_f^{RLQG}) is called as the *robust LQG controller*. ■

In the case of no uncertainty, that is, $M = N = 0$, the robust LQG controller becomes the nominal LQG controller. The robust LQG controller guarantees the robust stability for all the allowable uncertainty and confines the LQG cost index, $J_{\Delta A}$, which depends on the uncertainty, to a certain bound. Such a concept for designing the robust controllers is referred to as the guaranteed cost controllers or the robust H_2 controllers (Bernstein and Haddad, 1989; Petersen, 1995).

It is noted that the Riccati solution matrix P in (11) is an implicit function of the scaling matrix as well as the controller gains. Therefore, the inclusion of scalings enlarges the feasible controller set for the robust LQG controller so that the upper bound of the cost index in (12) can be made to be tight. In the most of literatures which deal with real parameter uncertainty, the scaling matrix is not included or used in the limited fashion. Moreover, scalings is not treated as a design variable but selected by the trial and error method in the step of determining the I/O decomposition (e. g. Bernstein and Haddad, 1988, 1989; Jabbari and Schmitendorf, 1993).

In the practical point of view, the robust LQG controller suffers from the numerical difficulty because the constraint (11) is nonlinear with respect to the variables. Even the feasibility can not be globally checked with the current numerical algorithms such as LMI method. Such a numerical difficulty is well known in the area of the robust H_2 control theory (Khou *et al.*, 1994) and developing the solving methods remains an open problem. In this note, the gradient search with Lagrange multiplier (Bazaraa et al., 1993), which is one of nonlinear programming techniques and commonly used in many nonlinear optimization problems, is applied for solving the robust LQG controller.

4. Numerical Example

Consider a flexible structure modeled by a five-mass system connected by four uncertain springs as shown in Fig. 1. The variations of spring constants imply the perturbation of the natural frequencies. We assumed the spring constants can vary within $\pm 5\%$ from the nominal values. The force actuator is located on the first mass and only the position of the fifth mass is measured. State vector is defined as $x^T = [x_1, x_2, x_3, x_4, x_5, \dot{x}_1, \dot{x}_2, \dot{x}_3, \dot{x}_4, \dot{x}_5]$. The used LQG parameters are given as $Q = diag[0, 0, 0, 0, 1, 0, 0, 0, 0, 0]$, $R = 0.01$, $F = B$, $W = 1$ and $V = 10$. To construct an I/O decomposition in (4b), we used the singular value decomposition method such that $E_i = U_i \sigma_i V_i^T = (U_i \sqrt{\sigma_i}) \cdot (\sqrt{\sigma_i} V_i^T) = M_i N_i$ by a MATLAB function svd. Since $rank(E_i) = 1$ for $i = 1, \cdots, 4$, the uncertainty and the scaling matrix have the following structure

$$\Delta(t) = diag[\delta_1,\ \delta_2,\ \delta_3,\ \delta_4],$$
$$\Gamma = diag[\gamma_1,\ \gamma_2,\ \gamma_3,\ \gamma_4],\ (\gamma_i \neq 0)$$

By using a gradient search with Lagrange multiplier, we solved the nonlinear optimization problem in Definition 3.1 and obtained the cost bound such that $J_{\Delta A} \leq 138.6$. The obtained robust LQG controller and the nominal LQG controller are as follows:

$$K_c^{RLQG} = \begin{bmatrix} 4.8093 \\ -5.3590 \\ 1.3041 \\ -0.8688 \\ 0.9946 \\ 2.5265 \\ 2.2706 \\ -0.3433 \\ 1.0144 \\ 0.4977 \end{bmatrix}^T \quad K_f^{RLQG} = \begin{bmatrix} -0.0040 \\ 1.4694 \\ 0.0585 \\ 0.1594 \\ 1.5671 \\ 0.0873 \\ -1.3098 \\ 2.4092 \\ -2.3647 \\ 1.4715 \end{bmatrix}$$

$$K_c^{LQG} = \begin{bmatrix} 7.3759 \\ 3.0828 \\ -0.5755 \\ 0.1494 \\ -0.0327 \\ 3.8408 \\ 9.8055 \\ 10.1419 \\ 9.9402 \\ 10.0162 \end{bmatrix}^T, \quad K_f^{LQG} = \begin{bmatrix} 0.3490 \\ 0.3578 \\ 0.3834 \\ 0.4396 \\ 0.6572 \\ 0.0113 \\ 0.0144 \\ 0.0256 \\ 0.0497 \\ 0.2159 \end{bmatrix}$$

In fact, we could not find any feasible set of gains by conventional approaches (Bernstein and Haddad, 1989; Petersen, 1995) because we should repeat the laborious modification of the I/O decomposition by the trial and error method. Figure 2 shows the quadratic performance

$$J_{\delta_2}(K_c, K_f) \equiv \lim_{t_f \to \infty} E\left[\frac{1}{t_f}\int_0^{t_f}(x^TQx + u^TRu)\,dt\right] \tag{14}$$

obtained by the given controller (K_c, K_f), with respect to the variation of δ_2 in the case of $\delta_1 = \delta_3 = \delta_4 = 0$. The nominal LQG controller cannot guarantee the robust stability for the uncertainty while the quadratic performance of the robust LQG controller is insensitive to the parameter variation. Figure 3 shows the control performances of two controllers in the presence of parameter uncertainties. Consequently, we can observe the robustness of our approach to parameter uncertainties. Figure 4 shows the sensitivity and complementary sensitivity functions defined as

$$S(s) = \{I + P(s)K(s)\}^{-1} \text{ and } T(s) = \{I + P(s)K(s)\}^{-1}P(s)K(s) \tag{15}$$

where $P(\bullet)$ and $K(\bullet)$ denote the nominal plant and the controller, respectively. As can be seen in Fig. 4(b), the robust LQG controller makes the notches, which correspond to the natural frequencies, deeper than that of LQG controller.

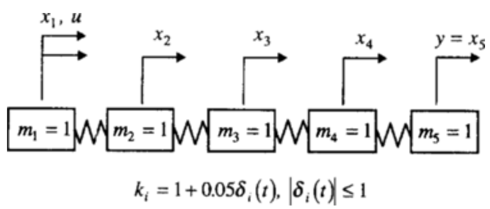


Fig. 1 A five-mass system connected by four uncertain springs.

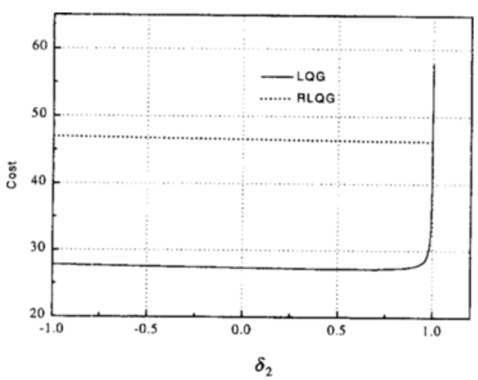
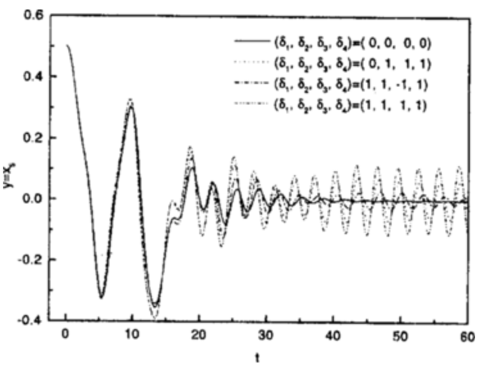
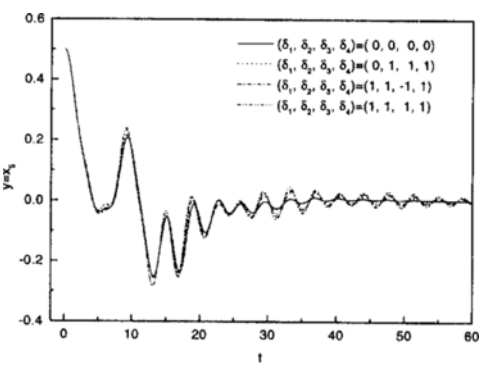


Fig. 2 Quadratic performance index $J_{\delta_2}(K_c, K_f)$ in the case of $\delta_1 = \delta_3 = \delta_4 = 0$.

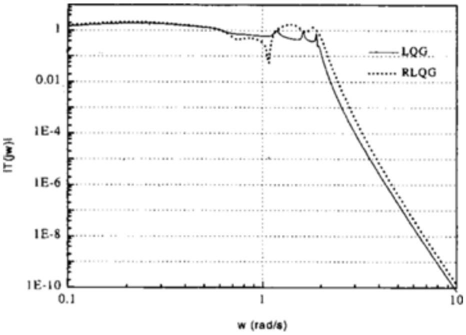


(a) Performance of LQG controller

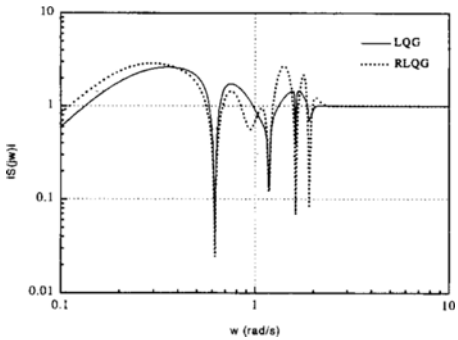


(b) Performance of robust LQG controller

Fig. 3 Regulating performances of the nominal LQG and the robust LQG controllers.



(a) Complementary sensitivity function



(b) Sensitivity function

Fig. 4 Sensitivity and Complementary sensitivity functions for the nominal system.

5. Concluding Remarks

In this note, we have proposed the robust LQG control design method for treating lightly damped systems with multiple natural frequency uncertainties. Since such an uncertainty is described by the form of real parameter uncertainty, which contains multiple uncertain parameters with multiple rank structure, the conventional quadratic stabilization technique can not be directly applied. The conventional methods can be improved by adopting the general I/O decomposition of the real parameter uncertainty. It is noted that the conventional quadratic stability leads to a block-diagonally scaled small gain condition, which makes it possible to treat the uncertainty of interest, with the help of the general I/O decomposition. The effectiveness of our approach was shown by a design example.

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